Bifibrations of Polycategories and Classical Linear Logic

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Abstract

The main goal of this article is to expose and relate different ways of interpreting the multiplicative fragment of classical linear logic in polycategories. Polycategories are known to give rise to models of classical linear logic in so-called representable $*$-polycategories, which ask for the existence of various polymaps satisfying the different universal properties needed to define tensor, par, and negation. We begin by explaining how these different universal properties can all be seen as instances of a single notion of universality of a polymap parameterised by an input or output object, which also generalises the classical notion of universal multimap in a multicategory. We then proceed to introduce a definition of in-cartesian and out-cartesian polymaps relative to a refinement system (= strict functor) of polycategories, in such a way that universal polymaps can be understood as a special case. In particular, we obtain that a polycategory is a representable $*$-polycategory if and only if it is bifibred over the terminal polycategory $\mathbb{1}$. Finally, we present a Grothendieck correspondence between bifibrations of polycategories and pseudofunctors into **MAdj**, the 2-polycategory of multivariable adjunctions. When restricted to bifibrations over $\mathbb{1}$ we get back the correspondence between $*$-autonomous categories and Frobenius pseudomonoids in **MAdj** that was recently observed by Shulman.

1 Introduction

In this paper we explore the theory of polycategorical bifibrations and their close connection to models of MLL the multiplicative fragment of classical linear logic. Our approach builds on and unifies two lines on work, polycategorical semantics of linear logic and fibrations of multicategories. The study of polycategories as models of MLL and their link to $*$-autonomous categories is possible through the notion of *two-tensor polycategory with duals* or *representable $*$-polycategory*. This goes back to Szabo's definition of polycategories in [Sza75] and was later revisited by Cockett and Seely [CS97], see also [BCST96, Hyl02, CS07]. On the other hand, This relation between $*$-autonomous categories and representable $*$-polycategories is analogous to the relation between monoidal categories and *representable multicategories* (called *monoidal multicategories* by Lambek [Lam69]), a relation studied carefully by Hermida [Her00]. Hermida noted certain analogies between the theory of representable multicategories and the theory of fibred categories (cf. [Her00, Table 1]), which he later made explicit by introducing a notion of (covariant) *fibration of multicategories* [Her04], in such a way that a representable multicategory is precisely the same thing as a multicategory fibred over the terminal multicategory $\mathbb{1}$. One interest of studying the more general notion of covariant fibration of multicategories $\mathcal{E} \to \mathcal{B}$, where every multimorphism $f : A_1, \ldots, A_n \to B$ in $\mathcal{B}$ induces a pushforward functor $\mathbf{push}\langle f \rangle : \mathcal{E}_{A_1} \times \cdots \times \mathcal{E}_{A_n} \to \mathcal{E}_B$, is that it models a much richer class of structures coming from algebra and logic. For example, Hermida notes that an *algebra for an operad O* can be identified with a discrete covariant

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fibration over O, the latter seen as a one-object multicategory. The appropriate definition of *contravariant* fibration (and of bifibration) of multicategories was not addressed in [Her04]. However, there is a natural definition of contravariant fibration of multicategories, made explicit in the work of Hörmann [HÏ7, A.2] and of Licata, Shulman, and Riley [LSR17], under which each multimorphism of the base multicategory induces a family of pullback operations $\mathbf{pull}[f]^{(i)} : \mathcal{E}^{\mathrm{op}}_{A_1} \times \cdots \times \mathcal{E}^{\mathrm{op}}_{A_{i-1}} \times \mathcal{E}^{\mathrm{op}}_{A_{i+1}} \times \cdots \times \mathcal{E}^{\mathrm{op}}_{A_n} \times \mathcal{E}_B \to \mathcal{E}_{A_i}$, parameterised by the selection of the index $1 \leq i \leq n$ of a particular input object A_i. One interesting feature of this definition is that *monoidal biclosed categories* in the sense of Lambek [Lam69] are equivalent to multicategories bifibred over $\mathbb{1}$.

The theory of bifibrations of polycategories that we present is guided by the principle that representable $*$-polycategories (and hence $*$-autonomous categories) should be equivalent to polycategories bifibred over the terminal polycategory $\mathbb{1}$. One consequence of this theory is that we recover a nice observation recently made by Shulman [Shu19], that $*$-autonomous categories are equivalent to (pseudo) Frobenius monoids in the (2-)polycategory of multivariable adjunctions. This is obtained a result of a general Grothendieck construction for bifibrations of polycategories, in a similar manner to the pattern mentioned above. Perhaps surprisingly, another one of our motivational examples for developing this theory was the category $\mathbf{FBan_1}$ of finite dimensional Banach spaces and contractive maps. It is $*$-autonomous and it comes with a forgetful functor into **FVect** that preserves this structure. However, contrary to the latter it is not compact closed. It provides a model of classical MALL based on finite dimensional vector spaces that is not degenerate, in the sense that the positive and negative fragments do not coincide. The fact that this category and more generally $\mathbf{Ban_1}$ the category of arbitrary Banach spaces and contractive maps provides a model of intuitionistic MALL is well-documented (cf. [BPS94]). However we could not find any mention of the fact that it is $*$-autonomous. Yet the structures needed to interpret the connectives are popular in the study of Banach spaces. For both $\otimes$ and $⅋$ these correspond to the tensor product of vector spaces but equipped with different norms called the projective and the injective (cross)norms. These have the property of being extremal in all the well-behaved norms that can be put on the tensor product. More specifically for any crossnorm $\| - \|$ and any $u \in A \otimes B$ we have $\|u\|_{A ⅋ B} \leq \|u\| \leq \|u\|_{A \otimes B}$. We will see that this is explained by the fact that the projective ($\otimes$) norm and the injective ($⅋$) norm are pushforwards and pullbacks, respectively, relative to the forgetful functor into vector spaces.

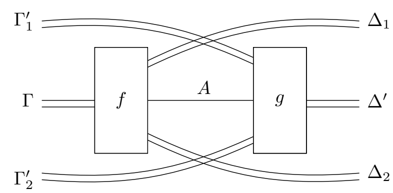
2 Note to the ACT reviewers

This document is an extended abstract for a paper that has been accepted to MFPS 2020 [BZ20]. We think that it would be of interest to the ACT community because of the structures it is concerned with - e.g. polycategories, Grothendieck construction, Frobenius monoids - and the examples considered - e.g. (normed) vector spaces, causal structures as in [KU17]. In the future, in addition to expanding the theory, we would like to find more examples that could benefit from the perspective developed in the paper. We are confident that the diversity of the ACT community would be the ideal setting to get ideas for potential further works in addition to general feedback.

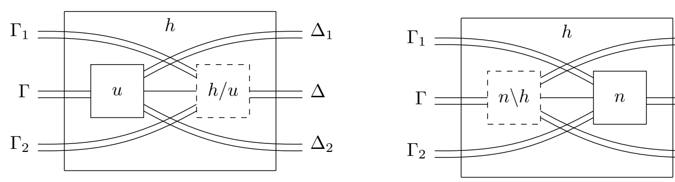
3 Content of the paper

Before giving more details, it is worth noticing that we do not assume that our polycategories are symmetric. In particular, the representable $*$-polycategories that we consider have both left and right duals and correspond to planar non-symmetric $*$-autonomous categories. In this extended abstract we will often skim over this subtleties since all the ideas described in the paper can be transferred to the symmetric setting almost straightforwardly.

Polycategories are a generalisation of categories and multicategories where the polymaps are many-to-many. Composition is done along one object, much like a cut rule in logic. We will depict it diagrammatically:



In the non-symmetric case there is a planarity condition on the contexts that ensures that wires don't cross. Furthermore there are some unitality and associativity conditions plus interchange laws that say that plugging multiple polymorphisms on the inputs or outputs of another one can be done in any order. Any $*$-autonomous category gives rise to a $*$-polycategory whose polymaps $f : A_1, ..., A_m \to B_1, ..., B_n$ correspond to maps $\mathcal{P}(f) : A_1 \otimes ... \otimes A_m \to B_1 \parr ... \parr B_n$. It is possible to characterise the polycategories coming from a $*$-autonomous category. In [CS97] this is done by asking for the existence of objects $A \otimes B$, $\top$, $A \parr B$, $\bot$ and A^* coming with polymaps having certain properties. The authors call such a polycategory a two-tensor polycategory with duals. Instead we take another approach that emphasises the connection with fibrations. We call a map $u : \Gamma \to \Delta_1, A, \Delta_2$ universal in the output A if any polymap of type $\Gamma_1, \Gamma, \Gamma_2 \to \Delta_1, \Delta, \Delta_2$ factors uniquely through u. Dually we defined what it means for a map to be universal in one of its input. Graphically,

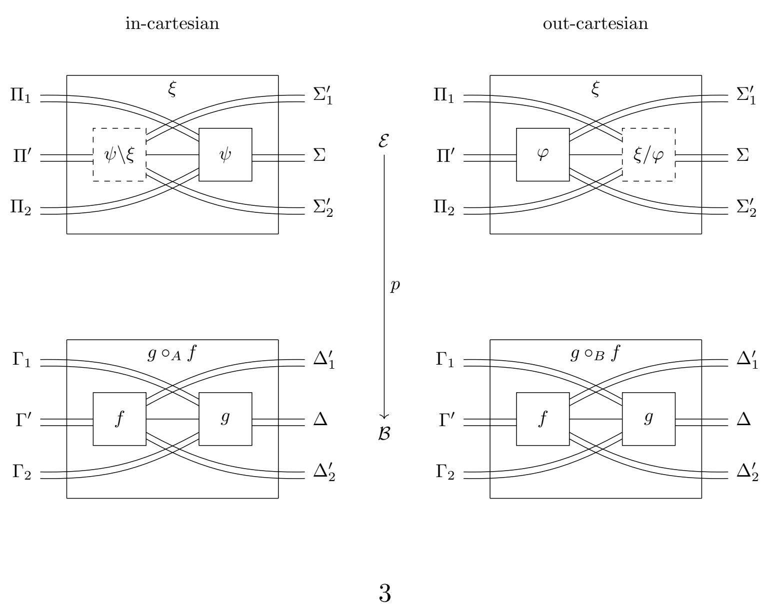


$(*)$

A $*$-representable polycategory is a polycategory such that for any contexts $\Gamma, \Delta_1, \Delta_2$ there is an object A and a polymap $\Gamma \to \Delta_1, A, \Delta_2$ universal in A and similarly for the input case. These two notions are equivalent.

Theorem 1. *$\mathcal{P}$ is a two-tensor polycategory with duals iff it is $*$-representable.*

After that we embark in the theory of bifibrations of polycategories. For a functor between polycategories $p : \mathcal{E} \to \mathcal{B}$ we define what it means for a polymaps to be cartesian in one of its inputs (*in-cartesian*) or outputs (*out-cartesian*) – these are analogous to the usual notions of cartesian and opcartesian maps in a fibred category. They should satisfy some factorisation property summarised diagrammatically below:



$(\dagger)$

We can see that this is similar to the definition of universal polymaps. Indeed when the base polycategory $\mathcal{B}$ is the terminal polycategory a polymap is in-cartesian (resp. out-cartesian) in R iff it is universal in R. In fact we establish an equivalence between $*$-autonomous categories and bifibrations over the terminal polycategory.

Theorem 2. *There is an equivalence between planar $*$-autonomous categories and bifibrations over the terminal polycategory $\mathbb{1}$.*

We get as a corollary that for a functor $p : \mathcal{E} \to \mathcal{B}$ such that $\mathcal{B}$ is $*$-representable, $\mathcal{E}$ is $*$-representable if it has all in-cartesian liftings of in-universal polymaps and all out-cartesian liftings of out-universal polymaps. We use this to lift the logical structure from **FVect** to $\mathbf{FBan_1}$. Instead of putting a global $*$-autonomous structure on the total category, we could want to give a local $*$-autonomous structure in the fibre over an object. We explain how to do that on the fibre over a Frobenius monoid by lifting the multiplication, comultiplication, unit and counit polymaps.

Finally, we develop a polycategorical "Grothendieck correspondence". It is inspired by the generalised Grothendieck correspondences going back to the work of Bénabou and explained for example in [B0́0]. First we give a correspondence between functors of polycategories $p : \mathcal{E} \to \mathcal{B}$ and lax normal functors $\mathcal{B} \to \mathbf{Dist}$ where **Dist** is the weak 2-polycategory of categories and distributors (a.k.a. profunctors). We then describe the link between the fibrational properties of p and representability of the distributors. This lets us refine the correspondence to connect bifibration of polycategories with pseudofunctors into **MAdj** the 2-polycategory of categories and multivariable adjunctions. In particular we can take the base polycategory to be the terminal polycategory. We know that bifibrations are then equivalent to $*$-autonomous categories. On the other hand a pseudofunctor $\mathbb{1} \to \mathbf{MAdj}$ is the same thing as a Frobenius pseudomonoid in **MAdj**. We recover a result recently advertised by Shulman [Shu19] that $*$-autonomous categories are the same as Frobenius pseudomonoids in **MAdj**.

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A Definitions and theorems

In this appendix we give for reference the formal definitions of some of the important notions that appear in the paper. We also sketch the proof of theorem 1.

Definition 1. *A* polycategory $\mathcal{P}$ *consists of:*

- *a collection of* objects $Ob(\mathcal{P})$
- *for any pair of finite lists of objects* Γ *and* Δ*, a set* $\mathcal{P}(\Gamma;\Delta)$ *of* polymaps *from* Γ *to* Δ *denoted* $f\colon \Gamma \to \Delta$ *(we refer to objects in* Γ *as* inputs *of* f*, and to objects in* Δ *as* outputs*)*
- *for every object* A*, an* identity *polymap* $id_A : A \to A$
- *for any pair of polymaps* $f\colon \Gamma \to \Delta_1, A, \Delta_2$ *and* $g\colon \Gamma'_1, A, \Gamma'_2 \to \Delta'$ *satisfying the restriction that [either* Δ_1 *or* Γ'_1 *is empty] and [either* Δ_2 *or* Γ'_2 *is empty], a polymap* $g \circ_A f\colon \Gamma'_1, \Gamma, \Gamma'_2 \to \Delta_1, \Delta', \Delta_2$

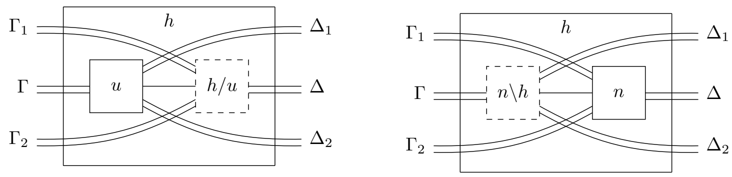
subject to appropriate unitality, associativity*, and* interchange *laws whenever these make sense:*

$$id_A \circ_A f = f \tag{1}$$
$$f \circ_A id_A = f \tag{2}$$
$$(h \circ_B g) \circ_A f = h \circ_B (g \circ_A f) \tag{3}$$
$$(h \circ_B g) \circ_A f = (h \circ_A f) \circ_B g \tag{4}$$
$$h \circ_B (g \circ_A f) = g \circ_A (h \circ_B f) \tag{5}$$

Definition 2. *A polymap* $u : \Gamma \to \Delta_1, A, \Delta_2$ *is said to be* universal in the output A *(or* out-universal *for short, or simply* universal *when there is no ambiguity), written* $u : \Gamma \to \Delta_1, \underline{A}, \Delta_2$ *if for any polymap* $h : \Gamma_1, \Gamma, \Gamma_2 \to \Delta_1, \Delta, \Delta_2$ *such that* $\Gamma_i = \emptyset$ *or* $\Delta_i = \emptyset$*, there is a unique polymap* $h/u : \Gamma_1, A, \Gamma_2 \to \Delta$ *such that* $h = h/u \circ_A u$*.*

Dually, a polymap $n : \Gamma_1, A, \Gamma_2 \to \Delta$ *is* universal in the input A *(or* in-universal*), written* $n : \Gamma_1, \underline{A}, \Gamma_2 \to \Delta$ *if for any polymap* $h : \Gamma_1, \Gamma, \Gamma_2 \to \Delta_1, \Delta, \Delta_2$ *such that* $\Gamma_i = \emptyset$ *or* $\Delta_i = \emptyset$ *there is a unique polymap* $n\backslash h : \Gamma \to \Delta_1, A, \Delta_2$ *such that* $h = n \circ_A n\backslash h$*.*

Graphically, the definitions are summarised in the following diagram:

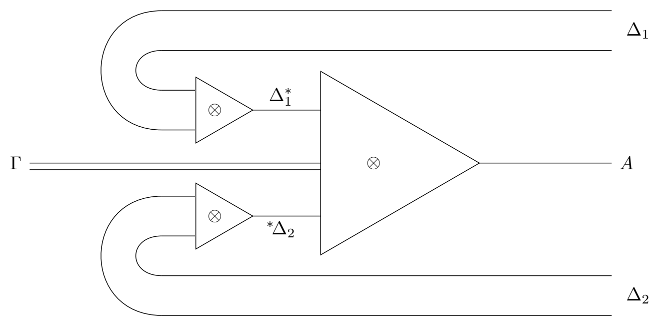


$(*)$

Definition 3. *A polycategory is said to be* $*$-representable *if it has all in-universal and out-universal objects, that is, if for any* Γ, Δ_1, Δ_2 *there is an object* A *equipped with an out-universal polymap* $\Gamma \to \Delta_1, \underline{A}, \Delta_2$, *and similarly, for any* Γ_1, Γ_2, Δ *there is an object* A *equipped with an in-universal polymap* $\Gamma_1, \underline{A}, \Gamma_2 \to \Delta$.

Theorem 3. *$\mathcal{P}$ is a two-tensor polycategory with duals iff it is $*$-representable.*

Proof. We prove that a $*$-representable polycategory is equivalent to a two-tensor polycategory with duals. The idea is that we get $\otimes$ by the polymap $A, B \to A \otimes B$ universal in $A \otimes B$, $⅋$ by the polymap $A ⅋ B \to A, B$ universal in $A ⅋ B$ and A^* by the polymap $\to A, A^*$ universal in A^*. Conversely, from $\otimes, ⅋, (-)^*$, for any $\Gamma, \Delta_1, \Delta_2$ we can define a universal polymap $\Gamma \to \Delta_1, A, \Delta_2$. It is represented by the following diagram:



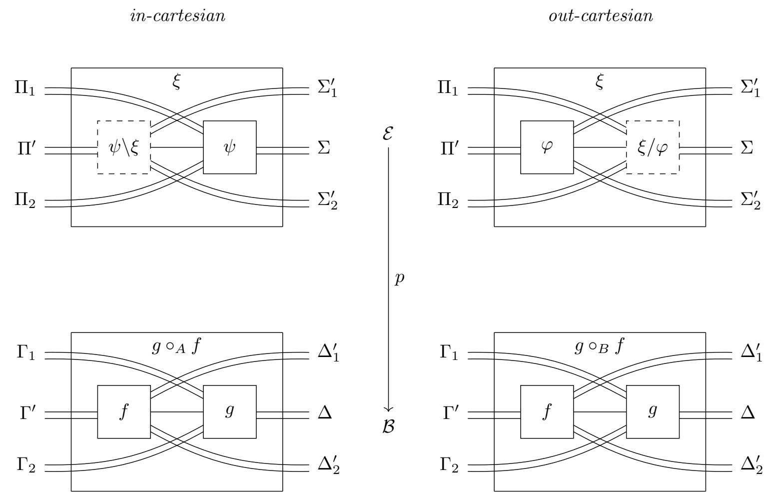
□

Definition 4. *A* poly-refinement system *is defined as a strict functor of polycategories* $p : \mathcal{E} \to \mathcal{B}$. *Explicitly,* p *sends objects* $R \in \mathcal{E}$ *to objects* $p(R) \in \mathcal{B}$ *and polymaps* $\psi : R_1, ..., R_m \to S_1, ..., S_n$ *in* $\mathcal{E}$ *to polymaps* $p(f) : p(R_1), ..., p(R_m) \to p(S_1), ..., p(S_n)$ *in* $\mathcal{B}$ *in such a way that identities and composition are preserved strictly. We write* $R \sqsubset A$ *(pronounced "R refines A") to indicate that* $p(R) = A$, *and extend this to lists of objects in the obvious way, writing* $\Pi \sqsubset \Gamma$ *to indicate that* $\Pi = R_1, \ldots, R_n$ *and* $\Gamma = A_1, \ldots, A_n$ *for some* $R_1 \sqsubset A_1, \ldots, R_n \sqsubset A_n$. *Finally, we write* $\psi : \Pi \underset{f}{\Longrightarrow} \Sigma$ *to indicate that* ψ *is a polymap* $\Pi \to \Sigma$ *in* $\mathcal{E}$ *such that* $p(\psi) = f$, *with the implied constraint that* $f : \Gamma \to \Delta$ *where* $\Pi \sqsubset \Gamma$ *and* $\Sigma \sqsubset \Delta$.

Definition 5. *Fix a poly-refinement system* $p : \mathcal{E} \to \mathcal{B}$, *and let* $\psi : \Pi_1, R, \Pi_2 \underset{g}{\Longrightarrow} \Sigma$ *be a polymap in* $\mathcal{E}$, *with some given input object* $R \sqsubset A$. *Then* ψ *is said to be* in-cartesian in R *(relative to* p*), written* $\psi : \Pi_1, \underline{R}, \Pi_2 \underset{g}{\Longrightarrow} \Sigma$, *if for any other polymap* $\xi : \Pi_1, \Pi', \Pi_2 \underset{g \circ_A f}{\Longrightarrow} \Sigma_1', \Sigma, \Sigma_2'$ *there exists a unique polymap* $\psi \backslash \xi : \Pi' \underset{f}{\Longrightarrow} \Sigma_1', R, \Sigma_2'$ *such that* $\xi = \psi \circ_R (\psi \backslash \xi)$.

Dually, let $\varphi : \Pi \underset{f}{\Longrightarrow} \Sigma_1, S, \Sigma_2$ *be a polymap with some given output object* $S \sqsubset B$. *Then* φ *is said to be* out-cartesian in S, *written* $\varphi : \Pi \underset{f}{\Longrightarrow} \Sigma_1, \underline{S}, \Sigma_2$, *if for any polymap* $\xi : \Pi_1', \Pi, \Pi_2' \underset{g \circ_B f}{\Longrightarrow} \Sigma_1, \Sigma', \Sigma_2$ *there exists a unique polymap* $\xi / \varphi : \Pi_1', S, \Pi_2' \underset{g}{\Longrightarrow} \Sigma'$ *such that* $\xi = \xi / \varphi \circ_S \varphi$.

Graphically, the definitions are summarised in the following diagram:



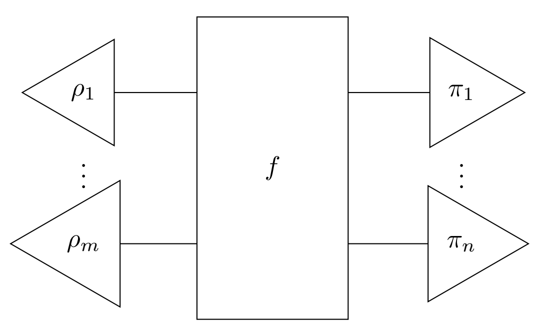
(†)

Definition 6. *A poly-refinement system $p : \mathcal{E} \to \mathcal{B}$ is said to be a* pull-fibration *if for any $f : \Gamma_1, A, \Gamma_2 \to \Delta$ in $\mathcal{B}$ and any $\Pi_1 \sqsubset \Gamma_1$, $\Pi_2 \sqsubset \Gamma_2$, and $\Sigma \sqsubset \Delta$ there is an object $\mathbf{pull}[f](\Pi_1 \smile \Pi_2; \Sigma) \sqsubset A$ together with an in-cartesian polymap $\Pi_1, \underline{\mathbf{pull}[f](\Pi_1 \smile \Pi_2; \Sigma)}, \Pi_2 \underset{f}{\Longrightarrow} \Sigma$. Dually, p is said to be a* push-fibration *if for any $f : \Gamma \to \Delta_1, B, \Delta_2$ in $\mathcal{B}$ and any $\Pi \sqsubset \Gamma$, $\Sigma_1 \sqsubset \Delta_1$, and $\Sigma_2 \sqsubset \Delta_2$ there is an object $\mathbf{push}\langle f\rangle(\Pi; \Sigma_1 \smile \Sigma_2) \sqsubset B$ together with an out-cartesian polymap $\Pi \underset{f}{\Longrightarrow} \Sigma_1, \underline{\mathbf{push}\langle f\rangle(\Pi; \Sigma_1 \smile \Sigma_2)}, \Sigma_2$. Finally, p is said to be a* bifibration *if it is both a pull-fibration and a push-fibration.*

B An example of bifibration

The forgetful functor $\mathbf{FBan_1} \to \mathbf{FVect}$ and the one $Caus(\mathcal{C}) \to \mathcal{C}$ of [KU17] give functors of polycategories that are not bifibrations. But they have in-cartesian liftings (resp. out-cartesian liftings) of in-universal polymaps (resp. out-universal polymaps) which is enough to lift the $*$-representability of the base polycategory to the total one. However those two functors embed in a bifibration construction that we describe in this appendix. To keep it concise we skip the proofs.

For the rest of this section we fix a compact closed category that we will see as a polycategory $\mathcal{C}$. We will also fix a submonoid of its scalar $\mathcal{M} \subset \mathcal{C}(\cdot, \cdot)$. For an object A we will call states, or global elements, of A elements $\rho \in \mathcal{C}(\cdot, A)$, effects the elements of $\pi \in \mathcal{P}(A, \cdot)$ and scalars elements of $\mathcal{C}(\cdot, \cdot)$. We will write states of A $\rho : A$, effects $\pi : A^*$ and the scalar $1 := id_I$. For a polymap $f : A_1, ..., A_m \to B_1, ..., B_n$ and states $\rho_i : A_i$ and effects $\pi_j : B_j$ we will write $(\pi_1, ..., \pi_n)f(\rho_1, ..., \rho_m)$ for the following scalar :



More generally we will write $(h_1, ..., h_m)g(f_1, ..., f_n)$ for the composition of g with the f_i on input A_i and h_j on output B_j. We will often abbreviate it $(\overrightarrow{h})g(\overrightarrow{f})$.

Definition 7. *For an object $A \in \mathcal{C}$ and a set of states $c \in \mathcal{C}(\cdot, A)$ we denote $c^* \in \mathcal{C}(A, \cdot)$ the following set of effects*

$$c^* := \{\pi : A^* | \forall \rho \in c,\ \pi \circ \rho \in \mathcal{M}\}$$

Definition 8. *The polycategory $\mathcal{P}(\mathcal{C})$ as*

- *for objects pairs $\mathbf{A} := (A, c_{\mathbf{A}})$ of an object $A \in \mathcal{C}$ and a closed set of states $c_{\mathbf{A}} \subset \mathcal{C}(\cdot, A)$, i.e. such that $c_{\mathbf{A}}^{**} = c_{\mathbf{A}}$*
- *for polymaps $f : \mathbf{A}_1, ..., \mathbf{A}_m \to \mathbf{B}_1, ..., \mathbf{B}_n$, polymaps $f : A_1, ..., A_m \to B_1, ..., B_n$ in $\mathcal{C}$ such that the following property holds*

$$\forall \rho_i \in c_{\mathbf{A}_i}\ \forall \pi_j \in c^*_{\mathbf{B}_j}\ (\pi_1, ..., \pi_n) f(\rho_1, ..., \rho_m) \in \mathcal{M}$$

It is easy to see that this forms a polycategory. We will call contractive the maps that have the property above. We will note $\rho : \mathbf{A}$ for $\rho \in c_{\mathbf{A}}$ and similarly for effects.

Proposition 1. *$\mathcal{P}(\mathcal{C})$ is a polycategory.*

We have a notion of duality in $\mathcal{P}(\mathcal{C})$.

Definition 9. $\mathbf{A}^* := (A^*, c^*_{\mathbf{A}})$

Proposition 2. *For any set c, $c^{***} = c^*$.*

In the following we will consider the forgetful functor $\mathcal{P}(\mathcal{C}) \to \mathcal{C}$. It is a bifibration of polycategories.

Definition 10. *Given $g : \Gamma'_1, A, \Gamma'_2 \to \Delta'$ and families of sets $c_{\mathbf{\Gamma}'_1}, c_{\mathbf{\Gamma}'_2}, c_{\mathbf{\Delta}'}$, we define the set*

$$\mathbf{pull}[g](c_{\mathbf{\Gamma}'_1} \llcorner c_{\mathbf{\Gamma}'_2}; c_{\mathbf{\Delta}'}) := \{\rho : A \mid \forall \overrightarrow{\rho}'_1 : \mathbf{\Gamma}'_1, \overrightarrow{\rho}'_2 : \mathbf{\Gamma}'_2, \overrightarrow{\pi}' : (\mathbf{\Delta}')^*,\ (\overrightarrow{\pi}')g(\overrightarrow{\rho}'_1, \rho, \overrightarrow{\rho}'_2) \in \mathcal{M}\}$$

Then we define $c_{\mathbf{pull}[g](\mathbf{\Gamma}'_1 \llcorner \mathbf{\Gamma}'_2; \mathbf{\Delta}')} := (\mathbf{pull}[g](c_{\mathbf{\Gamma}'_1} \llcorner c_{\mathbf{\Gamma}'_2}; c_{\mathbf{\Delta}'}))^{**}$

Proposition 3. $\mathbf{pull}[g](c_{\mathbf{\Gamma}'_1} \llcorner c_{\mathbf{\Gamma}'_2}; c_{\mathbf{\Delta}'})$ *is closed so* $c_{\mathbf{pull}[g](\mathbf{\Gamma}'_1 \llcorner \mathbf{\Gamma}'_2; \mathbf{\Delta}')} = \mathbf{pull}[g](c_{\mathbf{\Gamma}'_1} \llcorner c_{\mathbf{\Gamma}'_2}; c_{\mathbf{\Delta}'})$

This is a pullback.

Proposition 4. *Given $g : \Gamma'_1, A, \Gamma'_2 \to \Delta'$ and families of sets $c_{\mathbf{\Gamma}'_1}, c_{\mathbf{\Gamma}'_2}, c_{\mathbf{\Delta}'}$, $\mathbf{push}\langle g \rangle(\mathbf{\Gamma}'_1; \mathbf{\Gamma}'_2 \llcorner \mathbf{\Delta}') := (A, c_{\mathbf{pull}[g](\mathbf{\Gamma}'_1 \llcorner \mathbf{\Gamma}'_2; \mathbf{\Delta}')})$ is a pullback.*

There is also a pushforward.

Definition 11. *Given $f : \Gamma \to \Delta_1, A, \Delta_2$ and families of closed sets $c_{\mathbf{\Gamma}}, c_{\mathbf{\Delta}_1}, c_{\mathbf{\Delta}_2}$ we define the following set*

$$\mathbf{push}\langle f \rangle(c_{\mathbf{\Gamma}}; c_{\mathbf{\Delta}_1} \llcorner c_{\mathbf{\Delta}_2}) := \{(\overrightarrow{\pi}_1, id_A, \overrightarrow{\pi}_2) f(\overrightarrow{\rho}) \mid \overrightarrow{\rho} : \mathbf{\Gamma}, \overrightarrow{\pi}_1 : \mathbf{\Delta}^*_1, \overrightarrow{\pi}_2 : \mathbf{\Delta}^*_2\}$$

We also define $c_{\mathbf{push}\langle f \rangle(\mathbf{\Gamma}; \mathbf{\Delta}_1 \llcorner \mathbf{\Delta}_2)} := \mathbf{push}\langle f \rangle(c_{\mathbf{\Gamma}}; c_{\mathbf{\Delta}_1} \llcorner c_{\mathbf{\Delta}_2})^{**}$

In this case the set $\mathbf{push}\langle f \rangle(c_{\mathbf{\Gamma}}; c_{\mathbf{\Delta}_1} \llcorner c_{\mathbf{\Delta}_2})$ was not closed so we really need to take its bidual.

Proposition 5. $\mathbf{push}\langle f \rangle(\mathbf{\Gamma}; \mathbf{\Delta}_1 \llcorner \mathbf{\Delta}_2) := (A, c_{\mathbf{push}\langle f \rangle(\mathbf{\Gamma}; \mathbf{\Delta}_1 \llcorner \mathbf{\Delta}_2)})$ *is a pushforward.*

This let us derive the $*$-representability of $\mathcal{P}(\mathcal{C})$.

Theorem 4. *$\mathcal{P}(\mathcal{C})$ is a representable $*$-polycategory with*

- $c_{\mathbf{A} \otimes \mathbf{B}} := \{\rho_A \otimes \rho_B \mid \rho_A : \mathbf{A}, \rho_B : \mathbf{B}\}^{**}$
- $c_{\mathbf{A} \parr \mathbf{B}} := \{\rho : A \otimes B \mid \forall \pi_A : \mathbf{A}^*, \forall \pi_B : \mathbf{B}^*,\ (\pi_A \otimes \pi_B)(\rho) \in \mathcal{M}\}$
- $c_{\mathbf{A}^*} := \{\pi : A^* \mid \forall \rho : \mathbf{A}\ \pi(\rho) \in \mathcal{M}$
- $c_{\mathbf{A} \multimap \mathbf{B}} := \{f : A \multimap B \mid \forall \rho_A : \mathbf{A}, \forall \pi_B : \mathbf{B}^*,\ (\pi_B) f(\rho_A) \in \mathcal{M}\}$

The construction $Caus(\mathcal{C})$ is obtained by asking for the existence of so called discard maps $A \to \cdot$ and some extra conditions on the sets c_A called flatness conditions. We get $\mathbf{FBan_1} \to \mathbf{FVect}$ by taking $\mathcal{C} := \mathbf{FVect}$ and restraining to the sets c_A that are the unit balls of norms.