Bifibrations of Polycategories and Classical Linear Logic[1]

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MFPS36, June 2020

[1] https://nicolas-blanco.github.io/publication/polybifibrations/

Outline

Categorical models of multiplicative linear logic

Intuitionistic MLL $(\otimes, 1, \multimap)$: monoidal closed categories

Classical MLL $(\otimes, 1, \parr, \bot, -^{*})$

- $*$-autonomous categories (Barr 1991)
- linearly distributive categories (Cockett and Seely 1997)

Multicategorical models of intuitionistic sequent calculus

An old idea[2]: replace categories by *multicategories*

- Multimaps $A_1, \ldots, A_n \to B$
- Inspired by linear algebra and sequent calculus (composition = cut)
- $\otimes, 1, \multimap$ can all be defined by *universal properties*

[2] Made explicit by Lambek (1969), and implicit in his earlier work.

Polycategorical models of classical sequent calculus

Another old idea: replace multicategories by *polycategories*

- Polymaps $A_1, \ldots, A_m \to B_1, \ldots, B_n$
- Originally used to model classical sequent calculus (Szabo 1975)
- Used to model MLL by Cockett and Seely, in a "two tensor polycategory with duals", a.k.a. *representable* $*$*-polycategory*

Fibrations of multicategories

Hermida (2000) observed several analogies between multicategory theory and fibred category theory. These analogies can be made rigorous via the notion(s) of *multicategorical fibration*. . .

Fibrations of multicategories

Covariant fibration of multicategories:

- Discussed by Hermida (2004)
- Monoidal category = multicategory fibred over $\mathbb{1}$
- Algebra for an operad $\mathcal{P}$ = discrete fibration over $\mathcal{P}$

Contravariant/bi-fibration of multicategories:

- Used in recent work of Licata, Shulman, and Riley (2017)
- Closely related to (bi)fibrations of (monoidal) closed categories
- Monoidal closed category = multicategory bifibred over $\mathbb{1}$

An important asymmetry: pullback along a multimap is parameterized by *a choice of input object*.

Our paper

Contribution #1: definition of *in-universal* and *out-universal* polymaps

- Notion of universality parameterized by an input or output object
- All of the MLL connectives (including negation) can be characterized by the existence of certain universal polymaps

Definition: a polycategory is $*$-*representable* if it has all universal polymaps

Theorem

$\mathcal{P}$ is $*$-representable iff it is a representable $*$-polycategory.

Our paper

Contribution #2: definition of *(bi)fibrations of polycategories*

- Notion of *in-cartesian* and *out-cartesian* polymaps, generalizing in-universal and out-universal
- $*$-autonomous category = polycategory bifibred over $\mathbb{1}$

One motivation: understand how compact closed structure on f-d vector spaces lifts to a $*$-autonomous structure on f-d Banach spaces, with $\otimes$ of vector spaces refined by the projective ($\otimes$) and injective ($\invamp$) crossnorms.

Our paper

Contribution #3: *polycategorical Grothendieck correspondences*

- A collection of Grothendieck correspondences for different classes of poly-refinement systems (functors of polycategories)
- The bifibrational case recovers Shulman's recent observation that $*$-autonomous categories are Frobenius pseudomonoids in **MAdj**, the 2-polycategory of multivariable adjunctions

Outline

1. Introduction
2. $*$-representable polycategories and universal polymaps
3. Bifibrations of polycategories
4. Conclusion
5. Bonus round: Polycategorical Grothendieck construction

Outline

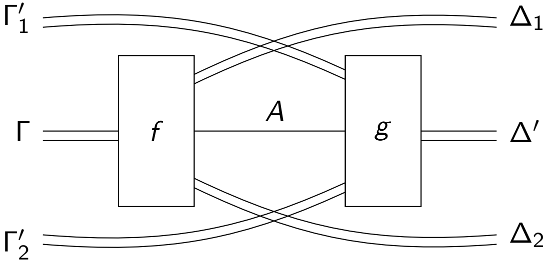
Polycategories

Definition (Polycategory)

A (planar) *polycategory* $\mathcal{P}$ has:

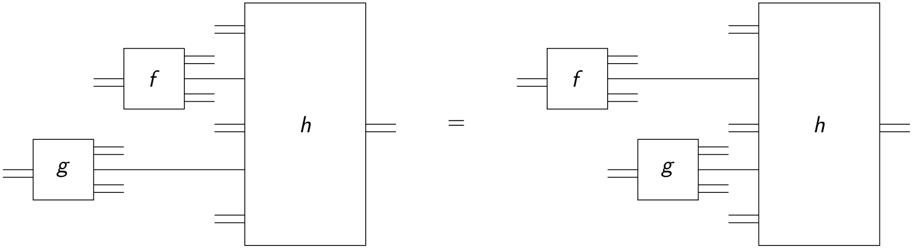
- A collection of objects
- For Γ, Δ finite lists of objects, a set of polymaps $\mathcal{P}(\Gamma; \Delta)$
- Identities $id_A : A \to A$
- Composition: $\dfrac{f : \Gamma \to \Delta_1, A, \Delta_2 \qquad g : \Gamma'_1, A, \Gamma'_2 \to \Delta'}{g \circ_A f : \Gamma'_1, \Gamma, \Gamma'_2 \to \Delta_1, \Delta', \Delta_2}$
- Planarity of $\circ$: $(\Gamma'_1 = \{\} \vee \Delta_1 = \{\}) \wedge (\Gamma'_2 = \{\} \vee \Delta_2 = \{\})$
- With unitality, associativity and two interchange laws

Composition of polymaps (graphically)



Planarity is also known as no-crossing condition (wires should not cross)

Interchange laws (graphically)



Examples

Example

There is a terminal polycategory $\mathbb{1}$. It has one object $*$ and a unique arrow $\overline{(m,n)} : *^m \to *^n$ for each arities m, n.

Example

Any linearly distributive category $\mathcal{C}$ (and $*$-autonomous category) gives a polycategory with polymaps $f : A_1 \otimes ... \otimes A_m \to B_1 \parr ... \parr B_n$.

Example

In particular any monoidal category gives a polycategory with the same objects and with polymaps $f : A_1 \otimes ... \otimes A_m \to B_1 \otimes ... \otimes B_n$.

Sequent calculus rules for MLL (fragment)

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, A \otimes B, \Gamma_2 \vdash \Delta} \otimes_L \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash \Delta, A \otimes B, \Delta'} \otimes_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad B, \Gamma' \vdash \Delta'}{\Gamma, A \parr B, \Gamma' \vdash \Delta, \Delta'} \parr_L \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, A \parr B, \Delta_2} \parr_R$$

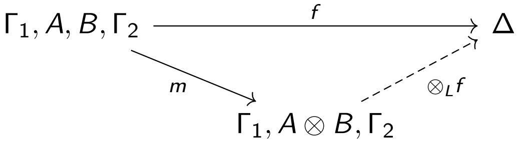
$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^*, \Delta} {}^*L \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^* \vdash \Delta} {}^*R$$

$\otimes$ in a polycategory

A tensor of A and B is an object $A \otimes B$ equipped with a binary map

$$m : A, B \to A \otimes B$$

with the following universal property:



Induces a natural isomorphism:

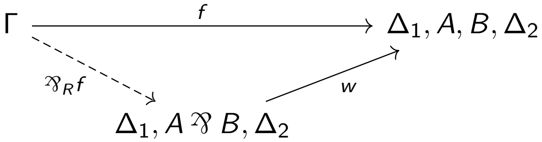
$$\mathcal{P}(\Gamma_1, A, B, \Gamma_2; \Delta) \simeq \mathcal{P}(\Gamma_1, A \otimes B, \Gamma_2; \Delta)$$

$\parr$ in a polycategory

A par of A and B is an object $A \parr B$ equipped with a co-binary map

$$w : A \parr B \to A, B$$

with the following universal property:



Induces a natural isomorphism:

$$\mathcal{P}(\Gamma; \Delta_1, A, B, \Delta_2) \simeq \mathcal{P}(\Gamma; \Delta_1, A \parr B, \Delta_2)$$

$-^*$ in a polycategory

A (right) dual of A is an object A^* equipped with polymaps

$$rcup_A : \cdot \to A, A^* \quad \text{and} \quad rcap_A : A^*, A \to \cdot$$

satisfying the equations ("snake identities"):

$$rcup_A \circ_{A^*} rcap_A = id_A \quad \text{and} \quad rcap_A \circ_A rcup_A = id_{A^*}$$

Induces a natural isomorphism:

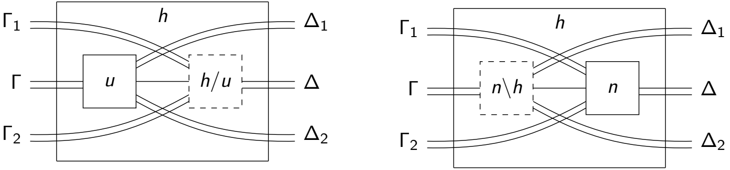
$$\mathcal{P}(\Gamma, A; \Delta) \simeq \mathcal{P}(\Gamma; A^*, \Delta)$$

A *representable $*$-polycategory* is a polycategory with all tensors, pars, left and right duals.

A parameterized notion of universality

Definition

- $u : \Gamma \to \Delta_1, A, \Delta_2$ *out-universal in* A, written $u : \Gamma \to \Delta_1, \underline{A}, \Delta_2$ if for any polymap $h : \Gamma_1, \Gamma, \Gamma_2 \to \Delta_1, \Delta, \Delta_2$, there is a unique polymap $h/u : \Gamma_1, A, \Gamma_2 \to \Delta$ such that $h = h/u \circ_A u$.
- $n : \Gamma_1, A, \Gamma_2 \to \Delta$ is *in-universal in* A, written $n : \Gamma_1, \underline{A}, \Gamma_2 \to \Delta$ if for any polymap $h : \Gamma_1, \Gamma, \Gamma_2 \to \Delta_1, \Delta, \Delta_2$ there is a unique polymap $n \backslash h : \Gamma \to \Delta_1, A, \Delta_2$ such that $h = n \circ_A n \backslash h$.



$(*)$

∗-representable polycategories

Definition

A polycategory is ∗-*representable* if it has all in-universal and out-universal objects, in the sense that:

- for any $\Gamma_1, \Gamma_2, \Delta$ there is an object A and a polymap $\Gamma_1, \underline{A}, \Gamma_2 \to \Delta$
- for any $\Gamma, \Delta_1, \Delta_2$ there is an object A and a polymap $\Gamma \to \Delta_1, \underline{A}, \Delta_2$

Theorem

$\mathcal{P}$ is a representable ∗-polycategory iff it is ∗-representable.

$*$-representable $\Rightarrow$ representable $*$-polycategory

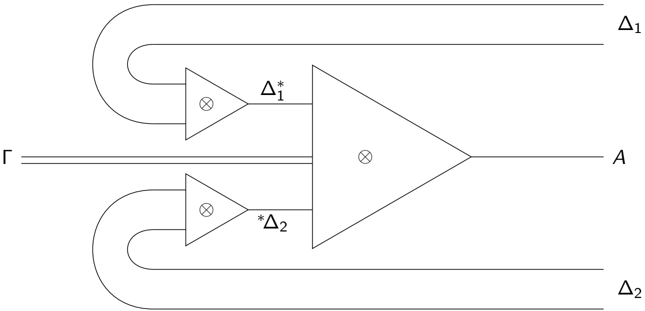
- Tensor is out-universal: $m_{\otimes} : \Gamma \to \underline{\bigotimes \Gamma}$
- Par is in-universal: $w_{\parr} : \underline{\bigparr \Delta} \to \Delta$
- Right dual is out-universal: $rcup_A : \cdot \to A, \underline{A^*}$
- Right dual is also in-universal $rcap_A : \underline{A^*}, A \to \cdot$

Remark

$rcup_A$ and $rcap_A$ are also universal in A

representable ∗-polycategory ⇒ ∗-representable

For $\Gamma, \Delta_1, \Delta_2$ we get an out-universal polymap $\Gamma \to \Delta_1, \underline{A}, \Delta_2$:



Outline

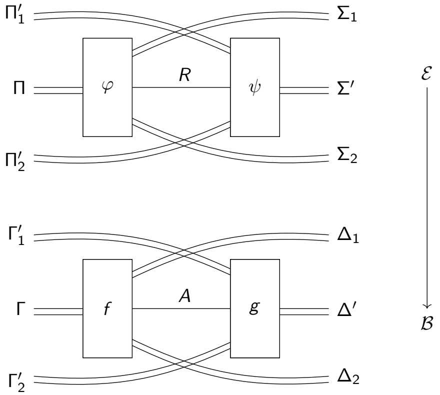
Poly-refinement systems

A *poly-refinement system* is a (strict) functor of polycategories $p : \mathcal{E} \to \mathcal{B}$.

(Terminology inspired from study of type refinement systems.)

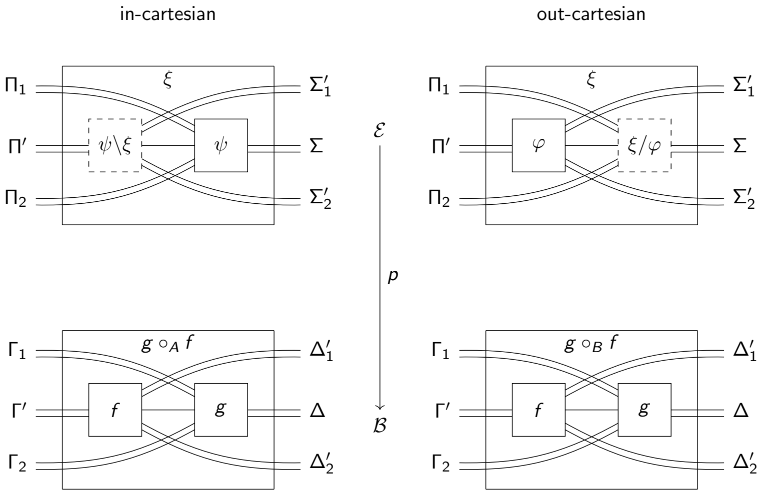
Poly-refinement systems (graphically)

Vertically, with the top diagram living in $\mathcal{E}$ and the bottom diagram in $\mathcal{B}$ in such a way that an object and polymaps are directly above their image. For example preservation of composition is given by:



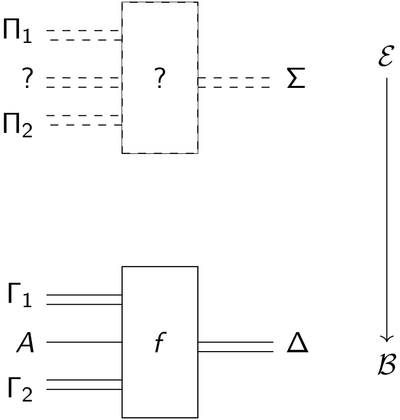
Cartesian polymaps (graphically)

Graphically, the definitions are summarized in the following diagram:



(†)

Cartesian liftings (example)



Polycategorical (bi)fibrations

Fix a poly-refinement system $p : \mathcal{E} \to \mathcal{B}$

Definition

- p is a *pull-fibration* if it has all in-cartesian liftings.
- p is a *push-fibration* if it has all out-cartesian liftings.
- p is a *bifibration* if it is both a pull-fibration and push-fibration.

$*$-representable polycategory = bifibration over $\mathbb{1}$

The notion of (in/out-)cartesian polymap immediately generalizes that of (in/out-)universal polymap.

In particular, a polycategory $\mathcal{P}$ is $*$-representable iff the terminal poly-refinement system $! : \mathcal{P} \to \mathbb{1}$ is a bifibration.

As a corollary, a $*$-autonomous category is “just” a polycategory fibred over the terminal polycategory!

Application: lifting of $*$-autonomous structure

Even if $p : \mathcal{E} \to \mathcal{B}$ is not a bifibration, it may have *some* cartesian liftings.
It is interesting to consider liftings of universal polymaps.

Proposition

In-cartesian liftings of in-universal polymaps are in-universal.
Out-cartesian liftings of out-universal polymaps are out-universal.

Corollary

If $\mathcal{B}$ is $*$-representable and $\mathcal{E}$ has all cartesian liftings of universal polymaps then $\mathcal{E}$ is $*$-representable.

Example: Banach spaces

Example

There are polycategories **Vect** and **FVect** of (finite dimensional) vector spaces and polylinear maps.

Example

There are polycategories $\mathbf{Ban_1}, \mathbf{FBan_1}$ of (finite dimensional) Banach spaces and contractive (norm-non-increasing) polylinear maps.

$f : A_1, \ldots, A_m \to B_1, \ldots, B_n$ contractive: $|(\varphi_1, \ldots, \varphi_n) f(a_1, \ldots, a_m)| \leq \prod_{i,j} \|a_i\|_{A_i} \|\varphi_j\|_{B_j^*}$

Example: Banach spaces

Forgetful functor: $\mathbf{FBan_1} \to \mathbf{FVect}$

Proposition

$\mathbf{FBan_1}$ has all cartesian liftings of universal polymaps.
So it is $*$-representable.

Proposition

$\|u\|_{A\otimes B} = \inf_{u=\sum_i a_i \otimes b_i} \sum_{i,j} \|a_i\|_A \|b_i\|_B$ the projective crossnorm

$\|u\|_{A⅋B} = \sup_{\|\varphi\|_{A^*},\|\psi\|_{B^*}\leq 1} |(\varphi \otimes \psi)(u)|$ the injective crossnorm

Outline

Conclusion

We have presented a fibrational perspective on models of classical MLL:

- $*$-autonomous categories as polycategories bifibred over $\mathbb{1}$
- used to construct liftings of $*$-autonomous structure along a functor

In the paper we also discuss a collection of polycategorical Grothendieck correspondences, and relate this to Shulman's recent analysis of $*$-autonomous categories as Frobenius pseudomonoids in **MAdj**.

Future work:

- Finding other interesting examples of polycategorical bifibrations
- Building polarized models
- Specialising to the base polycategory being compact closed

Summary table

classical MLL	$\otimes, 1$	$⅋, \bot$	$*$
***-autonomous category**	monoidal structure	monoidal structure	monoidal duality
Representable polycategory	out-universal objects	in-universal objects	in-universal/out-universal object
Bifibred polycategory	pushforwards	pullbacks	pullback/pushforward
Frobenius pseudomonoid in MAdj	multiplication + unit	comultiplication + counit	unit/co-unit + adj

Table: Models of classical MLL

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Outline

Distributors

Warning

In the following 2-polycategories are weak when strictness is not explicitly mentioned. We will assume some of the theory of 2-polycategories.

Definition

There is a 2-polycategory **Dist** induced by the compact closed structure of the bicategory of distributors/profunctors/modules.

Multivariable adjunctions

Definition

A *multivariable adjunction* is a distributor that is representable in each of its variable.

Example: multivariable adjunction $A, B \to C, D$

Four functors:

- $f_A : C \times D \times B^{\mathrm{op}} \to A$
- $f_B : A^{\mathrm{op}} \times C \times D \to B$
- $f_C : A \times B \times D^{\mathrm{op}} \to C$
- $f_D : C^{\mathrm{op}} \times A \times B \to D$

such that for any $a \in A, b \in B, c \in C, d \in D$
$A(a, f_A(c, d, b)) \simeq B(b, f_B(a, c, d)) \simeq C(c, f_C(a, b, d)) \simeq D(d, f_D(c, a, b))$

MAdj

Definition

There is a 2-polycategory **MAdj** with polymaps the multivariable adjunctions.

Proposition

MAdj is a $*$-polycategory with duality $(-)^* = (-)^{\mathrm{op}}$.

Example

- A $(1,1)$-adjunction $A \to B$ is an adjunction.
- A $(0,1)$-adjunction $\cdot \to A$ is an object of A (representable presheaf)
- A $(1,0)$-adjunction $A \to \cdot$ is an object of A (representable copresheaf)
- A $(0,0)$-adjunction is a set

Fibres of a poly-refinement system

$p : \mathcal{E} \to \mathcal{B}$ a poly-refinement system

Proposition

There is a lax normal functor $\partial p : \mathcal{B}^{\mathrm{op}} \to \mathbf{Dist}$ sending an object to its fibre and a morphism f to the distributor $\partial p(f)$ consisting of the sets $\partial p(f)(\Pi; \Sigma) = \{\varphi : \Pi \underset{f}{\Longrightarrow} \Sigma\}$ acted on by precomposition and postcomposition.

$$\partial p(f)(\Pi_1, -, \Pi_2; \Sigma) = Hom(-, \mathbf{pull}[f](\Pi_1 \mathbin{\llcorner\!\lrcorner} \Pi_2; \Sigma))$$
$$\partial p(f)(\Pi; \Sigma_1, -, \Sigma_2) = Hom(\mathbf{push}\langle f\rangle(\Pi; \Sigma_1 \mathbin{\llcorner\!\lrcorner} \Sigma_2), -)$$

Proposition

If p bifibration then this is a pseudofunctor $\partial p : \mathcal{B}^{\mathrm{op}} \to \mathbf{MAdj}$.

Polycategorical Grothendieck construction

$F : \mathcal{B}^{\mathrm{op}} \to \mathbf{Dist}$ lax normal functor

Definition

$\int F$ is the polycategory with objects pairs (A, R) with $A \in \mathcal{B}$ and $R \in F(A)$ and polymaps $(f, \varphi) : (\Gamma, \Pi) \to (\Delta, \Sigma)$ with $f \in \mathcal{B}$ and $\varphi \in F(f)(\Pi, \Sigma)$. There is a first projection functor $\pi_1 : \int F \to \mathcal{B}$.

$F : \mathcal{B}^{\mathrm{op}} \to \mathbf{MAdj}$ pseudofunctor

Proposition

$\pi_1 : \int F \to \mathcal{B}$ is a bifibration.

Polycategorical Grothendieck correspondence

The construction ∂ and $\int$ are inverse to each other.

Theorem

This establishes correspondences between

- Poly-refinement system $\mathcal{E} \to \mathcal{B}$ and lax normal functors $\mathcal{B}^{\mathrm{op}} \to$ **Dist**
- Bifibrations $\mathcal{E} \to \mathcal{B}$ and pseudofunctors $\mathcal{B}^{\mathrm{op}} \to$ **MAdj**

Frobenius pseudomonoid

Definition

Frobenius pseudomonoid in 2-polycategory $\mathcal{P}$: object A equipped with unique polymaps $\overline{(m,n)_A} : A^m \to A^n$ for each $m, n \in \mathbb{N}$ such that $\overline{(1,1)_A} \simeq id_A$ and these polymaps are stable under composition up to iso.

Proposition

Equivalently a Frobenius pseudomonoid is a pseudofunctor $\mathbb{1} \to \mathcal{P}$.

Theorem (Shulman)

There is a correspondence between Frobenius pseudomonoid in **MAdj** *and* $*$*-autonomous categories.*